MSO2120 Coursework 1 (2020/21)

Total Marks: 50

Number of questions: 10

Attempt all questions.

Unless told otherwise, Theorems from the course may be used without proof.

Any theorems which are used must be explicitly stated.

You may lose marks for giving a long answer where there is a short one.

You must submit your answers as **one pdf file** on the course **MyLearning page** by:

23:59 on **Friday 20th November 2020**.

«Student_number»

Coursework 1

1. Prove that the following limits hold directly from the definition of a limit:

   a.
$$\lim_{n\to\infty} «c1» n^{-«c2»} = 0$$

[2 marks]

   b.
$$\lim_{n\to\infty} \frac{«c3» n}{«c4» - n} = -«c3»$$

[3 marks]

2.
   a. Prove the following: If a sequence (a_n) converges to a and (b_n) converges to b then the sequence $\frac{a_n}{b_n}$ converges to $\frac{a}{b}$, provided that $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$.

[5 marks]

   b. Using the result of part a, evaluate the following limit:
$$\lim_{n\to\infty} \frac{«c5» n^5 - 3n^2 + n}{2n^5 - 15n^2 + 2n - «c6»}$$
ensuring that the steps in your calculations are rigorous and clear.

[3 marks]

   c. Evaluate the following limit
$$\lim_{n\to\infty} -n + \sqrt{n^2 + «c7» n}$$
and prove that the sequence converges to this limit.

[3 marks]

3. Find the upper and lower limits of the following sequences and justify that the values you have found are the upper and lower limits:

   a. $a_n = \sin^2\left(\frac{n\pi}{4}\right), n \in \mathbb{N};$

[3 marks]

   b. $b_n = (-1)^{\mathrm{n}} \frac{(«c8» + n)}{7 - «c9» n}, n \in \mathbb{N}.$

[3 marks]

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4. A sequence (a_n) is defined to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists N such that:

$$|a_n - a_m| < \varepsilon$$

for all $n, m \geq N$.

   a. Prove that if a sequence (a_n) converges then it is a Cauchy sequence.

[3 marks]

   b. Is every Cauchy sequence bounded? Justify your claim.

[3 marks]

5.

   a. Does the following series converge? Prove your claim.

$$\sum_{n=1}^{\infty} \frac{(«c10»)^n}{(2n+1)!}$$

   You may use results without proof from the course but you must state them clearly.

[2 marks]

   b. Does the following series converge? Prove your claim.

$$\sum_{n=1}^{\infty} \left(\frac{«c11»n - «c12»n^7}{«c13»n^7 + 4} \right)^n$$

   You may use results without proof from the course but you must state them clearly.

[2 marks]

6. Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(«c15»)^n} (x + «c14»)^n$$

   a. Find the radius of convergence for this series stating any result you use.

[4 marks]

   b. Determine whether the series converges or diverges at the endpoints of that interval stating clearly any results which you have used.

[4 marks]

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7. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^{«c16»}$. Evaluate

$$\lim_{x \to 0} f(x)$$

and prove your claim directly from the definition of the limit of a function.

[2 marks]

8. A function $f: D \to \mathbb{R}$ is said to be **Lipschitz continuous** if there is a constant $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in D$.

Show that the function $f(x) = \frac{1}{x}$ is Lipschitz continuous on [1,2].

[3 marks]

9. Let the polynomial $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = ax^{«c17»} + bx^{«c18»} + cx + d$$

for constants $a, b, c, d \in \mathbb{R}$.
Prove, stating clearly any results you use from the course, that the function is continuous everywhere.

[2 marks]

10. Prove, stating clearly any results you use from the course, that the function $g: \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x^{«c19»}\sin\left(\frac{1}{x}\right) + x^{«c20»} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous everywhere.

[3 marks]